

### Special Case

If  $A$  is a triangular matrix, then its determinant is the product of the entries on the main diagonal

$$\text{Ex. } A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{pmatrix} \quad \det(A) = 2 \times -2 \times 1 \times 3 = -12$$

### Evaluation of a determinant using elementary operations

Let  $A$  and  $B$  be square matrices

- ① If  $B$  is obtained from  $A$  by interchanging 2 rows of  $A$ , then  $\det(B) = -\det(A)$

$$\text{eg. } A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \quad \det(A) = -6 \\ \det(B) = 6$$

- ② If  $B$  is obtained from  $A$  by adding a multiple of row of  $A$  to another row of  $A$  then  $\det(B) = \det(A)$

- ③ If  $B$  is obtained from  $A$  by multiplying a row of  $A$  by a non zero constant ( $c$ ) then  $\det(B) = c \det(A)$

$$\text{Find } \det(A), \quad A = \begin{pmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{pmatrix} \\ \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & 2 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \\ |A| = (-7)(1 \times -1 \times -1) = -7$$

\* 7 is taken as common factor



Ex Find  $\det(A)$   $A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{pmatrix}$   $\xrightarrow{\substack{R_2 - 4R_1 \\ R_3 + 2R_1 \\ R_4 - 3R_1}} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$\det(A) = (1 \times 2 \times -3 \times -1) = 6$

### Determinants and elementary column operations

Theorem, If  $A$  is an  $n \times n$  matrix then  $\det(A^T) = \det(A)$

\* Rules are the same as the previous row rules

Ex evaluate

$\begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix} \quad \det = -(1 \times 4 \times 2) = -8$

Ex Find  $|A|$

$A = \begin{pmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{pmatrix} \xrightarrow[\text{Common factor from } C_2]{\text{take } -2} \begin{pmatrix} -1 & -1 & 2 \\ 3 & 3 & 4 \\ 5 & 5 & -3 \end{pmatrix} \xrightarrow{C_2 - C_1} \begin{pmatrix} -1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & -3 \end{pmatrix}$

$\det(A) = \text{Zero}$

Conditions that produce a zero determinant

- ① An entire row/column consists of zeros
- ② Two rows/columns are equal
- ③ One row/column is a multiple of another row/column



Ex. Evaluate  $|A|$

$$A = \begin{pmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{pmatrix} \xrightarrow[R_5 - R_2]{R_4 + R_2} \begin{pmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ -3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= 1 \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{C_1 - 3C_4} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \xrightarrow{R_1 + R_2} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} = 5(-8 \times 5) - (-1 \times 13) = -135$$

### properties of determinants

① If  $A$  and  $B$  are square matrices of order  $n$  then  $\det(AB) = \det(A) \cdot \det(B)$

Note:  $\det(A+B) \neq \det(A) + \det(B)$

② If  $A$  is an  $n \times n$  matrix and  $(c)$  is a scalar then  $\det(cA) = c^n \det(A)$

$$\text{eg } A = \begin{pmatrix} 30 & 20 & 50 \\ 80 & 40 & 30 \\ 20 & 10 & 60 \end{pmatrix}_{3 \times 3} \quad A = 10 \begin{pmatrix} 3 & 2 & 5 \\ 8 & 4 & 3 \\ 2 & 1 & 6 \end{pmatrix}, \quad \begin{vmatrix} 3 & 2 & 5 \\ 8 & 4 & 3 \\ 2 & 1 & 6 \end{vmatrix} = -21$$

$$\det A = 10^3 \times -21 = -21000$$

③ A square matrix is invertible if and only if its determinant  $\neq 0$



④ If  $A$  is invertible, then  

$$\det(A^{-1}) = \frac{1}{\det A}$$

Proof

$$AA^{-1} = I$$

$$|AA^{-1}| = |I|$$

$$|A| \cdot |A^{-1}| = 1$$

$$|A^{-1}| = \frac{1}{|A|}$$

⑤  $\det(A^T) = \det(A)$

Equivalent Conditions for a non-singular matrix

If  $A$  is non singular then:

1-  $A$  is invertible

2-  $\det(A) \neq 0$

3-  $Ax = b$  has a unique solution

4-  $Ax = 0$  has only the trivial solution (where variables are all zeros)

Ex Which of the following has unique solution

a)  $2x_2 - x_3 = 1$

$$3x_1 - 2x_2 + x_3 = 4$$

$$3x_1 + 2x_2 - x_3 = -4$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix}$$

$$C_3 = \frac{C_2}{-2} \Rightarrow \det = 0$$

does not have unique solution

b)  $2x_2 - x_3 = -1$

$$3x_1 - 2x_2 + x_3 = 4$$

$$3x_1 + 2x_2 + x_3 = -4$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\det \neq 0$$

has unique solution



## Applications of determinants

### 1- Finding inverse of a matrix

$$A^{-1} = \frac{1}{|A|} \cdot \text{adjoint}(A)$$

$$\text{Adjoint matrix} = (\text{cofactor matrix})^T$$

Ex Find the inverse of A

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 6 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$\text{Cofactor matrix}(A) = \begin{pmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{pmatrix} \quad \text{Adjoint matrix}(A) = \begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\det |A| = 3$$

$$A^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{pmatrix}$$

### 2- Cramer's Rule

#### Theorem

The unique solution of  $Ax = b$  has entries given by  $x_i = \frac{\det A_i(b)}{\det(A)}$

Where  $A_i(b)$  is the matrix obtained from A by replacing column i by the vector b



Ex Find  $x_1$  given the following system

$$5x_1 + x_2 - x_3 = 4$$

$$9x_1 + x_2 - x_3 = 1$$

$$x_1 - x_2 + 5x_3 = 2$$

$$A = \begin{pmatrix} 5 & 1 & -1 \\ 9 & 1 & -1 \\ 1 & -1 & 5 \end{pmatrix} \quad \det(A) = -16$$

$$A_1 = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & -1 & 5 \end{pmatrix} \quad \text{first column in } A \text{ is replaced by the column of } b$$

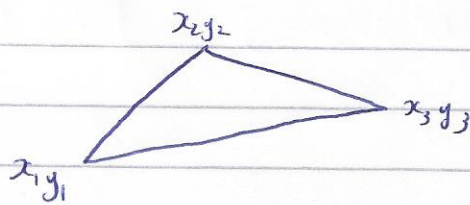
$$\det A_1 = 12$$

$$x_1 = \frac{12}{-16} = -\frac{3}{4}$$

### Applications of determinants in geometry

#### 1. Area of a triangle in a plane

$$\text{Area} = \pm \frac{1}{2} \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}$$



\*  $\ominus$  sign is used to change the sign of determinant if it was -ve

#### 2. Test of collinearity in plane

$$\det = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \begin{matrix} (x_1, y_1) \\ (x_2, y_2) \\ (x_3, y_3) \end{matrix}$$

### 3. Equation of a line in plane

$(x_1, y_1), (x_2, y_2)$

$$\det = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \text{equation of straight line}$$

or  $y = mx + c$   
or  $ax + by = c$

### 4. Volume of tetrahedron

$$\text{Volume} = \pm \frac{1}{6} \det \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix}$$

